Covariance Based Uncertainty Analysis with Unscented Transformation

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Model is nonlinear

\[ y = f\left[ x \right] \]

**Known methods:**

1) Linearization (LIN)
2) Unscented Transformation (UT)
3) Higher Order UT (HOUT)
4) Monte-Carlo Simulation (MCS)
Unscented Transformation (1/2)

Sigma points:

\[ X_0 = m_x \]
\[ X_i = m_x + \gamma \cdot \left( \sqrt{V_x} \right)_i, \text{ for } i = 1, \ldots, n \]
\[ X_i = m_x - \gamma \cdot \left( \sqrt{V_x} \right)_{i-n}, \text{ for } i = n+1, \ldots, 2n \]
\[ \gamma = \sqrt{n + \lambda} \quad \lambda \text{ - scaling parameter} \]

The mean and covariance matrix:

\[ m_y = \sum_{i=0}^{2n} W_i^{(m)} \cdot Y_i \quad V_y = \sum_{i=0}^{2n} W_i^{(c)} \cdot (Y_i - m_y) \cdot (Y_i - m_y)^T \]
\[ W_0^{(m)} = \lambda/(n + \lambda) \quad W_0^{(c)} = \lambda/(n + \lambda) + (1 - \alpha^2 + \beta) \]
\[ W_i^{(m)} = W_i^{(c)} = 1/[2 \cdot (n + \lambda)], \quad i = 1, \ldots, 2n \]
Unscented Transformation (2/2)

Algorithm

\[
\begin{align*}
\mathbf{m}_X & \rightarrow \mathbf{V}_X \rightarrow \gamma \cdot \sqrt{\mathbf{V}_X} \rightarrow + \rightarrow f[\cdot] \rightarrow \{\mathbf{Y}_i\} \rightarrow \mathbf{W}_c \rightarrow \mathbf{m}_Y \\
\{\mathbf{X}_i\} & \rightarrow \{\mathbf{X}_i\} \rightarrow \mathbf{m}_X
\end{align*}
\]
Cartesian to Polar Coordinate Transformation (1/3)
Example 1

Known characteristics

\[ \mathbf{m}_x = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} \]
\[ \mathbf{V}_x = \begin{bmatrix} 0.005^2 & 0 \\ 0 & 0.005^2 \end{bmatrix} \]

Unknown characteristics

\[ \mathbf{m}_y = ? \]
\[ \mathbf{V}_y = ? \]

\[ \mathbf{x} = \begin{bmatrix} \text{Re} \\ \text{Im} \end{bmatrix} \quad \rightarrow \quad \mathbf{y} = \begin{bmatrix} A \\ \text{Ph} \end{bmatrix} = \begin{bmatrix} \sqrt{\text{Re}^2 + \text{Im}^2} \\ \text{atan} \left( \frac{\text{Im}}{\text{Re}} \right) \end{bmatrix} \]
Cartesian to Polar Coordinate Transformation (2/3)

Example 1
Cartesian to Polar Coordinate Transformation (3/3)

Example 1

Means of magnitude obtained by different methods

Result: linearization error depend on the average level of the desired signal

Standard uncertainties of magnitude obtained by different methods
Loss Comparison Method in Power Meter Calibration (1/2)

Example 2

Measured signal

\[ M = 1 - |\Gamma|^2 \]

Transformation

\[ x = \begin{bmatrix} \text{Re} \\ \text{Im} \end{bmatrix} \rightarrow y = M = 1 - \text{Re}^2 - \text{Im}^2 \]

The Jacobian

\[ J = \begin{bmatrix} -2 \text{Re} & -2 \text{Im} \end{bmatrix} \]

Linearization:

\[ m_x = \begin{bmatrix} m_{\text{Re}} \\ m_{\text{Im}} \end{bmatrix} \]

\[ m_y = 1 - m_{\text{Re}}^2 - m_{\text{Im}}^2 \]

\[ V_x = \begin{bmatrix} \sigma_{\text{Re}}^2 & 0 \\ 0 & \sigma_{\text{Im}}^2 \end{bmatrix} \rightarrow V_y = J \cdot V_x \cdot J^T = 4 \cdot \begin{bmatrix} \sigma_{\text{Re}}^2 m_{\text{Re}}^2 + \sigma_{\text{Im}}^2 m_{\text{Im}}^2 + 2\rho \sigma_{\text{Re}} \sigma_{\text{Im}} m_{\text{Re}} m_{\text{Im}} \end{bmatrix} \]
Loss Comparison Method in Power Meter Calibration (1/2)

Example 2

Comparison of the standard deviation (SD) obtained by different methods

\[ E_\sigma = 20 \cdot \log_{10} \left( \left| \frac{\sigma - \sigma_{\text{MCS}}}{\sigma_{\text{MCS}}} \right| \right) \]
1) One specific application of unscented transformations (UT) and high order unscented transformation (HOUT) is presented.

2) The UT and HOUT can predict the mean and covariance matrix with second or higher order accuracy, but do not require derivation of Jacobians or Hessians.

3) The UT is better suited than the linearization for the vector-network-analyzer uncertainty analysis at low signal levels.

4) For high signal level (or low noise), or linear models, the unscented transformations give the same results as linearization.

5) The performance benefits of the unscented transformation were demonstrated in a realistic example.
Thanks! Questions?

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References


